Uncertainties in the measurement of linear acceleration using laser interferometry

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ABSTRACT

In the International System of Units, SI, the linear acceleration quantity and the other dynamic quantities are defined as derived quantities. Depending upon both measurement standards and measurement methods they are traceable to certain basic quantities in the SI. The uncertainty of the measurement gives an idea about the length of the traceability chain up to the standards of basic quantities and the accuracy of the methods used in every calibration step. Experimental results of a Michelson interferometer, whose standards are traceable to basic units of length and time, are shown.

The experimental method analyzed in this paper is the fringe counting method. The uncertainty budget, which is based on the first order Taylor series approximation, expressed for the sensitivity calibration of a piezoelectric accelerometer, is analyzed. The obtained uncertainty model has the ability to identify the most important sources of uncertainty, besides further improvements to the measurement accuracy are discussed. Also, the models of some of the identified sources of uncertainty are experimentally validated and their results are shown.

The classical approach of the uncertainty, which uses only first order Taylor series approximation, is compared with an approximation using higher order terms in the Taylor series. Some concluding remarks about the model of uncertainty are made based on these two approaches. Experimental validation of both approximations are presented and discussed in the work.

NOMENCLATURE

S_C charge sensitivity of the accelerometer, pC/(m/s^2)
E charge amplifier output voltage, mV
λ laser He-Ne wavelength, nm
F_F photodiode output frequency, Hz
F_E excitation frequency on the accelerometer, Hz
A_C charge amplifier sensitivity, mV/pC
u_C combined standard uncertainty
r(xi,xj) correlation coefficient between xi and xj

1. INTRODUCTION

Nowadays, every measurement result has two parameters related to it, i.e., its mean value and its uncertainty. The mean value is the expected true value and the measurement uncertainty gives an idea about the total variability of the measurement process. The uncertainty could be thought as the "quality" of the measurement, it means that the higher measurement quality, the smaller related uncertainty.

The uncertainty of the measurement also gives an idea about the length of the traceability chain up to the standards of basic quantities and the accuracy of the methods used in every calibration step. The traceability chart for the charge sensitivity calibration of an accelerometer, when the fringe counting method is used with a Michelson interferometer, is shown below.

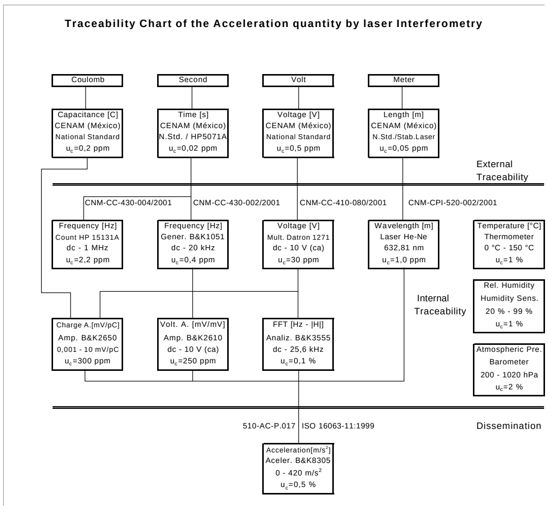

Figure 1

The reference that is known as 'GUM' or 'The Guide' [1], describes the process to estimate the uncertainty of measurement. The main aims of the GUM are: i. to promote full information on how uncertainty statements are arrived at, and; ii. to provide a basis for the comparison of measurement results. Therefore, if we need to 'compare' some measurement result, it is useful to take into account the uncertainties of each measurement process included in the comparison.

The starting point of every uncertainty budget is a mathematical model. This model is a functional relationship between the measurand, which is the specific quantity to be measured; and the input quantities, which are quantities whose values and uncertainties are measured directly in the experimental setup. The process to model the measurement is shown in the following section.

2. MODELING THE MEASUREMENT

In many experimental cases a measurand, y, is not measured directly, but is determined from N other quantities, x_1, x_2, . . . , x_N, through a functional relationship, f,

$$y = f(x_1,x_2, . . . ,x_N) \qquad (1)$$

Each input quantity, xi, and its associated standard uncertainty, $u(x_i)$, are obtained from a distribution of possible values, i.e., Gaussian, t-student, uniform. Besides, two types of uncertainty are defined, following a Bayesian standpoint. Type A uncertainties are evaluations of standard uncertainty components founded on frequency distributions, i.e., standard deviation of the mean. Type B uncertainties are evaluations founded on a priori distributions, i.e., specifications, standards, scientific research, etc.

Once all the independent sources of uncertainty are taken into account, they are combined based on a first order Taylor series approximation of y, the expression is the well known law of propagation of uncertainty, and can be expressed as follows,

$$\begin{aligned} u_C^2 &= \sum_{i=1}^{N}\left(\frac{\partial f}{\partial xi}\right)^2 u^2(xi) \\ &+ 2\sum_{i=1}^{N-1}\sum_{j=1}^{N}\frac{\partial f}{\partial xi}\frac{\partial f}{\partial xj}u(xi)u(xj)r(xi,xj) \end{aligned} \qquad (2)$$

where r(xi,xj)=r(xj,xi), and -1≤r(xi,xj)≤+1, is the correlation coefficient between xi and xj. The result of this law is the combined standard uncertainty, u_C, which is an estimated standard deviation and characterizes the dispersion of the values that could reasonably be attributed to the measurand, y.

Although u_C can be universally used to express the uncertainty of a measurement result, in some applications it is necessary to give a measure of uncertainty that defines an interval about the measurement result that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand. The additional measure of uncertainty that meets the requirement of providing an interval of the kind indicated above is termed expanded uncertainty and is denoted by U. The U is obtained by multiplying the combined standard uncertianty, u_C, by a converage factor, k,

$$U = k \cdot u_C(y) \qquad (3)$$

Then, the U can be interpreted as defining an interval about the measurement result that encompasses a large fraction p of the probability distribution characterized by that result and its combined standard uncertainty, and p is the level of confidence of the interval. To obtain the value of the coverage factor, k, that produces an interval corresponding to a specified level of confidence p, a t-distribution of the measurand, y, may be considered and the effective degrees of freedom, ν_{eff}, can be obtained from the Welch-Satterhwaite formula [1]

$$n_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{N}\frac{u_i^4(y)}{n_i}} \qquad (4)$$

In the following section, the estimation of the sensitivity magnitude of an accelerometer is described, when a calibration by laser interferometry is used. The calibration method is known as the fringe counting method.

3. CHARGE SENSITIVITY UNCERTAINTY

Several references [2,3] describe the sensitivity calibration of accelerometers by the fringe counting method. The measurand, which in this case is the charge sensitivity of the accelerometer, can be expressed by the next relationship,

$$S_C = \frac{2E}{p^2 l F_F F_E A_C} \qquad (5)$$

The relationships shown above are applied, as an example, for the calibration of the sensitivity magnitude of an accelerometer. The charge sensitivity of the accelerometer used for this example is 0,9931 pC/(m/s^2). This result is obtained when the calibration frequency is, F_E=159,155 Hz (1 000 rad/s) and the acceleration level is 50 m/s^2, approximately. Both the calibration frequency and the acceleration level are kept stable and constant during all the measurements included in the calibration process.

Once we have already defined the measurand, which in this case is the charge sensitivity of the accelerometer expressed by the equation 5, then the law of propagation of uncertianty, equation 2, can be applied to the expression of the measurand. This classical approach, which considers only first order terms in the Taylor series approximation is well known in metrology, from this approach we can obtain an expression for the combined standard uncertainty of the measurand, $u_C(S_C)$, as shown in the next equation,

$$
\begin{aligned}
u_C^2(S_C) = {} & \left(\frac{\partial S_C}{\partial E}\right)^2 u^2(E) + \left(\frac{\partial S_C}{\partial l}\right)^2 u^2(l) \\
& + \left(\frac{\partial S_C}{\partial F_E}\right)^2 u^2(F_E) + \left(\frac{\partial S_C}{\partial F_F}\right)^2 u^2(F_F) \\
& + \left(\frac{\partial S_C}{\partial A_C}\right)^2 u^2(A_C) \\
& + \frac{1}{2}\frac{\partial S_C}{\partial E}\frac{\partial S_C}{\partial F_F} u(E) u(F_F) r(E, F_F) \\
& + \frac{1}{2}\frac{\partial S_C}{\partial E}\frac{\partial S_C}{\partial F_E} u(E) u(F_E) r(E, F_E) \\
& + \frac{1}{2}\frac{\partial S_C}{\partial F_F}\frac{\partial S_C}{\partial F_E} u(F_F) u(F_E) r(F_F, F_E)
\end{aligned}
\tag{6}
$$

where the partial derivatives, ($\partial S_C/\partial x_N$), are called sensitivity coefficients. These sensitivity coefficients give the change in S_C, which is due to a unit change in each input quantity, x_N. As mentioned in section 2, by definition there are two type of uncertainties, i.e., Type A and B; every input quantity indicated in the expression of the measurand, i.e., E, λ, F_F, F_E, and A_C, may have either the two types of uncertainty or only one type. In the Table 1, at the end of this paper, the uncertainty budget for the sensitivity calibration is shown. By convolution the different types of uncertainties can be combined.

The expressions of the sensitivity coefficients, which are the partial derivatives of S_C with respect to each input quantity, are given below.

$$
\frac{\partial S_C}{\partial E} = \frac{2}{p^2 l F_F F_E A_C} = 2{,}51 \times 10^{-4} \left[\frac{\left(\dfrac{pC}{m/s^2}\right)}{mV} \right]
\tag{7}
$$

$$
\frac{\partial S_C}{\partial l} = -\frac{2E}{p^2 l^2 F_F F_E A_C} = -1{,}11 \times 10^{+6} \left[\frac{\left(\dfrac{pC}{m/s^2}\right)}{m} \right]
\tag{8}
$$

$$
\frac{\partial S_C}{\partial F_F} = -\frac{2E}{p^2 l F_F^2 F_E A_C} = -8{,}72 \times 10^{-6} \left[\left(\frac{pC}{m/s^2}\right) \cdot (s) \right]
\tag{9}
$$

$$
\frac{\partial S_C}{\partial F_E} = -\frac{2E}{p^2 l F_F F_E^2 A_C} = -4{,}41 \times 10^{-3} \left[\left(\frac{pC}{m/s^2}\right) \cdot (s) \right]
\tag{10}
$$

$$
\frac{\partial S_C}{\partial A_C} = -\frac{2E}{p^2 l F_F F_E A_C^2} = -7{,}04 \times 10^{-3} \left[\left(\frac{pC}{m/s^2}\right) \cdot \left(\frac{pC}{mV}\right) \right]
\tag{11}
$$

The correlation coefficient, r(xi,xj), characterizes the degree of correlation between xi and xj. Experimentally, the correlation coefficients that were not measured at the same time are zero, i.e., the charge amplifier sensitivity, A_C, and the laser He-Ne wavelength, λ. The input quantities mentioned were measured before the sensitivity calibration of the accelerometer. Therefore, the correlation coefficients, which are different than zero, are shown below,

$r(E\text{-}F_F) = +\,0{,}26$ (12)

$r(E\text{-}F_E) = +\,0{,}06$ (13)

$r(F_F\text{-}F_E) = -\,0{,}62$ (14)

The classical approach, which is based on first order Taylor series approximation, assumes that the model of the measurand is working in its very linear range, therefore the higher order terms can be ignored. However, to check if the expression for the charge sensitivity of the accelerometer is working in its linear range, we will use up to second order terms in the Taylor series approximation, obtaining the next relationship for the combined standard uncertainty,

$$
\begin{aligned}
u_C^2(S_C) = {} & \left(\frac{\partial S_C}{\partial E}\right)^2 u^2(E) + \left(\frac{\partial S_C}{\partial l}\right)^2 u^2(l) \\
& + \left(\frac{\partial S_C}{\partial F_F}\right)^2 u^2(F_F) + \left(\frac{\partial S_C}{\partial F_E}\right)^2 u^2(F_E) \\
& + \left(\frac{\partial S_C}{\partial A_C}\right)^2 u^2(A_C) + \frac{1}{4}\left(\frac{\partial^2 S_C}{\partial l^2}\right)^2 u^4(l) \\
& + \frac{1}{4}\left(\frac{\partial^2 S_C}{\partial F_F^2}\right)^2 u^4(F_F) + \frac{1}{4}\left(\frac{\partial^2 S_C}{\partial F_E^2}\right)^2 u^4(F_E) \\
& + \frac{1}{4}\left(\frac{\partial^2 S_C}{\partial A_C^2}\right)^2 u^4(A_C) \\
& + 2\frac{\partial S_C}{\partial E}\frac{\partial S_C}{\partial F_F} u(E) u(F_F) r(E, F_F) \\
& + 2\frac{\partial S_C}{\partial E}\frac{\partial S_C}{\partial F_E} u(E) u(F_E) r(E, F_E) \\
& + 2\frac{\partial S_C}{\partial F_E}\frac{\partial S_C}{\partial F_F} u(F_E) u(F_F) r(F_E, F_F) \\
& + \frac{\partial S_C}{\partial E}\frac{\partial^2 S_C}{\partial F_F^2} u(E) u^2(F_F) r(E, F_F) \\
& + \frac{\partial S_C}{\partial E}\frac{\partial^2 S_C}{\partial F_E^2} u(E) u^2(F_E) r(E, F_E) \\
& + \frac{\partial S_C}{\partial F_F}\frac{\partial^2 S_C}{\partial F_E^2} u(F_F) u^2(F_E) r(F_F, F_E) \\
& + \frac{\partial S_C}{\partial F_E}\frac{\partial^2 S_C}{\partial F_F^2} u(F_E) u^2(F_F) r(F_E, F_F) \\
& + \frac{\partial S_C}{\partial l}\frac{\partial^2 S_C}{\partial l^2} u^3(l) + \frac{\partial S_C}{\partial F_F}\frac{\partial^2 S_C}{\partial F_F^2} u^3(F_F) \\
& + \frac{\partial S_C}{\partial F_E}\frac{\partial^2 S_C}{\partial F_E^2} u^3(F_E) + \frac{\partial S_C}{\partial A_C}\frac{\partial^2 S_C}{\partial A_C^2} u^3(A_C) \\
& + \frac{1}{2}\frac{\partial^2 S_C}{\partial F_F^2}\frac{\partial^2 S_C}{\partial F_E^2} u^2(F_F) u^2(F_E) r(F_F, F_E)
\end{aligned}
\tag{15}
$$

where the partial derivatives, $(\partial S_C/\partial x_N)$, are the sensitivity coefficients, $u(x_N)$, are the standard uncertainties for each input quantity and, $r(xi,xj)$, are the correlation coefficients between xi and xj.

Looking at equation 15 it is easy to note that it has much more terms than equation 6. While 21 terms are needed for the Taylor series approximation using up to second order terms (eq. 15), only 8 terms are needed if a classic first order Taylor series approximation is used (eq. 6). However, the final combined standard uncertainty of the charge sensitivity using any of the two approximations does not have a significant difference, therefore it can be concluded that the expression of the charge sensitivity calibration by the fringe counting method is working in its very linear range.

4. CONCLUSIONS

As shown in section 1, the traceability chart gives information about the length of the traceability chain up to the standards of basic quantities and the accuracy of the methods used in every calibration step.

Once we have already defined a functional relationship of the measurand, which in this case is the charge sensitivity of the accelerometer expressed by the equation 5, then the law of propagation of uncertianty, equation 2, can be applied to the expression of the measurand. The classical approach considers a first order Taylor series approximation, this approach assumes that the model of the measurand is working in its very linear range.

To check if the expression for the charge sensitivity of the accelerometer is working in its linear range, up to second order terms in the Taylor series approximation were used. This approximation was compared with the classical approach, the results were $u_C^2(S_C) = 1{,}26 \times 10^{-8}$ for the classical approach and $u_C^2(S_C) = 1{,}23 \times 10^{-8}$ up to second order terms in the Taylor series approximation are used. The differences obtained by the two approaches have not practical significance, therefore the charge sensitivity model is working in its linear range and the higher order terms can be ignored.

REFERENCES

[1] *Guide to the Expression of Uncertainty in Measurement*, BIPM/ IEC/ IFCC/ ISO/ OIML/ IUPAC, 1995.

[2] Silva Pineda, G., Ferrer, L., *Estimation of uncertainties for the accelerometer calibration using laser interferometry*, SPIE, Vol. 4072, pp. 137-145, 2000.

[3] ISO 16063-11:1999, *Primary vibration calibration by laser interferometry*.

Table 1. Uncertainty budget (using a first order Taylor series approximation)

Quantity	Level (x_i)	Type / std uncert / Distribution	$u^2(x_i)$	$\left(\frac{\partial S_C}{\partial x_i}\right)^2$	Factor^2	Contribution
Voltage, E	2 803,238 *mV*	(A) 0,007 mV (B:Trac) 0,028 mV (B:Res) 0,006 mV	0,000 9 mV^2	6,28x10^{-8}	5,44x10^{-11}	0,4 %
Wavelength, λ	632,8 *nm*	(B:Trac) 3,3x10^{-14}m	1,09x10^{-27} m^2	1,23x10^{12}	1,34x10^{-15}	0,0 %
Beat frequency, F_F	80 498,929 *Hz*	(A) 10,042 Hz (B:Res) 5,8x10^{-4} Hz	100,851 Hz^2	7,61x10^{11}	7,67x10^{-9}	60,9 %
Excitation frequency, F_E	159,155 *Hz*	(A) 2,214x10^{-5} Hz (B:Res) 5,8x10^{-6} Hz	5,233x10^{-10} Hz^2	1,95x10^{-5}	1,02x10^{-14}	0,0 %
Charge amplifier sensitivity, A_C	99,7752 *mV/pC*	(B:Trac) 0,01 mV/pC	0,000 1 mV^2/pC^2	4,95x10^{-5}	4,95x10^{-9}	39,3 %

Correlation	1/2	$\frac{\partial S_C}{\partial x_i}$	$\frac{\partial S_C}{\partial x_i}$	$u(x_i)$	$u(x_j)$	$r(x_i,x_j)$		
$E - F_F$	0,5	2,51x10^{-4}	-8,72x10^{-6}	0,029 5 mV	10,042 4 Hz	0,26	-8,40x10^{-11}	-0,7 %
$E - F_E$	0,5	2,51x10^{-4}	-4,41x10^{-3}	0,029 5 mV	0,000 02 Hz	0,06	-2,23x10^{-14}	0,0 %
$F_F - F_E$	0,5	-8,72x10^{-6}	-4,41x10^{-3}	10,042 4 Hz	0,000 02 Hz	-0,62	-2,74x10^{-12}	0,0 %

Level of confidence, p =	**≈ 95 %**	Combined uncertainty squared	$u_c^2(S_C)$	1,26x10^{-8}	[pC/(m/s^2)]^2
Effective degrees of freedom, ν_{eff} =	**6**	Combined uncertainty	$u_c(S_C)$	0,000 11	pC/(m/s^2)
Coverage factor, k =	**2,65**	Expanded uncertainty in pC/(m/s^2)	$U(S_C)$	**0,000 29**	pC/(m/s^2)
		Expanded uncertainty in %	$U(S_C)$	**0,03**	%
		Charge sensitivity of the accelerometer	S_C	**0,9931**	pC/(m/s^2)